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LETTER TO THE EDITOR

Asymptotic spatial patterns on the complex time-dependent Ginzburg–Landau equation

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Abstract. Asymptotic spatial patterns (wavefront solutions) on time-dependent Ginzburg–Landau (TDGL) equations with complex coefficients are discussed. The condition for the existence of a spatial limit cycle solution is found to be given in terms of the propagation velocity of the wavefront solution and the stability criterion for the spatial limit cycle is obtained. The amplitude oscillation of an asymptotic spatial pattern on the TDGL equation (non-linear Schrödinger equation) with purely imaginary coefficients is expressed in terms of Jacobi's elliptic function, while an exact solution of the asymptotic spatial pattern on the TDGL equation (force-free Duffing equation) with purely real coefficients is aperiodic and has a unique propagation velocity.

Recently there has been extensive discussion on time-dependent Ginzburg–Landau (TDGL) equations with complex coefficients, which were derived in various fields of physics such as phase transition in non-equilibrium systems [1], instabilities in hydrodynamic systems [2], drift dissipative instability in plasma [3], chemical turbulence [4] and the distribution of gene frequency [5].

As was shown by Aronson and Weinberger [6], the solution of the TDGL equation converges to a wavefront (travelling wave) solution. Thus, TDGL equations reduce to ordinary equations. This gives us a generalisation of the Sagdeev potential [7] into a dissipative or non-potential system.

The condition for the existence of a spatial limit cycle solution is found to be given in terms of the propagation velocity of the wavefront solutions. We obtain a generalised (two-dimensional and dissipative) Mathieu equation which gives the stability criterion for a spatial limit cycle.

The formulation of the law of classical mechanics in terms of the Hamiltonian does not materially decrease the difficulty of solving any given problem. We work out practically the same differential equation as is provided by Newton's equation of motion. We then go on to invoke the Hamilton–Jacobi equation which provides an alternative method for integrating Newton's equation of motion. Thus in this letter we discuss asymptotic spatial patterns on the TDGL equation (non-linear Schrödinger equation) with purely imaginary coefficients, invoking the Hamilton–Jacobi equation.

An asymptotic spatial pattern on the non-linear Schrödinger equation corresponds to two-dimensional motion of a charged particle in a constant uniform magnetic field. The limiting amplitude oscillation of the above asymptotic pattern is given in terms of Jacobi's elliptic function.

Finally we discuss an asymptotic spatial pattern on the TDGL equation (force-free Duffing equation) with purely real coefficients. We can obtain an exact solution of an autonomous non-linear partial differential equation in the following way. First we find a scaling variable [8] from the linear part of the non-linear partial differential equation.

Next we seek a series solution of the given equation, supposing that the scaling variable is small. If the series solution converges, we obtain a desired exact solution. Even if the scaling variable is not small, the obtained exact solution is valid because of the continuity of the solution. The exact solution of the force-free Duffing equation is aperiodic and has a unique propagation velocity.

The TDGL equation with complex coefficients has the following form:

$$\frac{\partial}{\partial t}\phi = (1+\mathrm{i}a)\frac{\partial^2}{\partial x^2}\phi + (1+\mathrm{i}c)\phi - (1+\mathrm{i}d)|\phi|^2\phi \tag{1}$$

where ϕ is a physical quantity and a, c and d are real. Assuming a travelling wave solution $\phi(t, x) = \phi(x - bt)$ (b being a constant), we obtain

$$(1+a\mathrm{i})\ddot{\phi} + b\dot{\phi} + (1+c\mathrm{i})\phi + (1+\mathrm{i}d)|\phi^2|\phi = 0 \tag{2}$$

or

$$\begin{aligned} &(\ddot{R} - \dot{\theta}R^2) + a(2\dot{\theta}\dot{R} + \ddot{\theta}R) + b\dot{R} + R - R^3 = 0 \\ &(2\dot{\theta}\dot{R} + \ddot{\theta}R) + a(\ddot{R} - \dot{\theta}R^2) + b\dot{\theta}R + cR - dR^3 = 0 \end{aligned} \tag{3}$$

where $\dot{\phi} \equiv \mathrm{d}\phi/\mathrm{d}z (z \equiv x - bt)$, $R \equiv |\phi|$ and $\tan\theta \equiv \mathrm{Im}\,\phi/\mathrm{Re}\,\phi$.
Equation (3) has the following limit cycle solution:

$$\begin{aligned} &R_0^2 = 1 - \dot{\theta}^2 \\ &\dot{\theta}_0 = \frac{1}{2(a-d)}\{b \pm [b^2 + 4(a-d)(b-d)]^{1/2}\}. \end{aligned} \tag{4}$$

From equation (4), we obtain the condition for the existence of a spatial limit cycle solution:

$$(a-c)^2 \geqslant b^2 \geqslant 4(a-d)(d-c). \tag{5}$$

We now consider the stability of equation (4) to a small perturbation, providing that $a, b, c, d \gg 1$ (non-linear Schrödinger equation). Let us introduce the small perturbation $\delta\phi$ in the form

$$\phi(z) = R_0 \exp(\mathrm{i}\theta_0) + \delta\phi. \tag{6}$$

Linearising equation (2) with regard to $\delta\phi$ leads to

$$\delta\ddot{\phi} - \mathrm{i}b\delta\dot{\phi} + c\delta\phi - \mathrm{d}R_0^2[2\delta\phi + \exp(\mathrm{i}kz)\delta\phi^*] = 0 \tag{7}$$

where the asterisk denotes the complex conjugate and $k \equiv \dot{\theta}_0$.

Equation (7) is a generalised (complex and dissipative) Mathieu equation which gives the stability criterion for a limit cycle solution (4). We seek a solution of equation (7) in the form

$$\delta\phi \propto \psi(z)\exp(\mathrm{i}kz). \tag{8}$$

As usual, assuming $\mathrm{d}\psi/\mathrm{d}z \propto \lambda\psi$ and $|\lambda| \ll k$, we obtain from equations (7) and (8)

$$\lambda^2 = d^2 - \frac{(-ak^2 + kb + c - 2d)^2}{(2ka - b)^2}. \tag{9}$$

The stability condition for the limit a cycle solution is that

$$\lambda^2 \leqslant 0.$$

We next consider an amplitude oscillation which is described by equation (3). For $a, b, c, d \gg 1$, equation (3) corresponds to the non-linear Schrödinger equation

$$a(\ddot{R} - \dot{\theta}^2 R) + b\dot{\theta}R + cR - dR^3 = 0 \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(a\dot{\theta}R - bR^2) = 0. \tag{11}$$

Without loss of generality, we can assume that a is positive definite. In equations (10) and (11) we can regard b as an external constant uniform magnetic field.

The equations of motion (10) and (11) are derived from a Lagrangian

$$L = \tfrac{1}{2}a(\dot{R}^2 + R^2\dot{\theta}^2) - \tfrac{1}{2}b\dot{\theta}R^2 - (\tfrac{1}{2}c - \tfrac{1}{4}dR^2)R^2. \tag{12}$$

The generalised momenta associated with the coordinates R and θ are defined as

$$P_r = \partial L/\partial \dot{R} = a\dot{R} \tag{13}$$

$$P_\theta = \partial L/\partial \dot{\theta} = (a\dot{\theta} - b)R^2. \tag{14}$$

It is evident from equation (12) that angular momentum P_θ is conserved. From equations (12)-(14) we find the Hamiltonian

$$H = \frac{P_r^2}{2a} + \frac{1}{2a}(P_\theta/R + \tfrac{1}{2}bR)^2 - \tfrac{1}{2}R^2(c' - \tfrac{1}{2}dR^2) \tag{15}$$

where $c' \equiv c - b^2/8a$.

Introducing the action $S(\equiv \int \mathrm{d}z\, L)$, we have the Hamilton-Jacobi equation

$$\frac{\partial}{\partial z}S + \frac{1}{2a}\left(\frac{\partial S}{\partial R}\right)^2 + \frac{1}{2a}\left(\frac{1}{R}\frac{\partial S}{\partial \theta} + \tfrac{1}{2}bR\right)^2 + \tfrac{1}{2}R^2(c' - \tfrac{1}{2}dR^2) = 0. \tag{16}$$

Since the coordinate θ is a cycle, action S is given in the form

$$S = -E_0 + P_0\theta + (2a)^{1/2}\int \mathrm{d}R\left(E_0 - \frac{1}{2a}(P_0/R + \tfrac{1}{2}bR)^2 + \tfrac{1}{2}R^2(c' - \tfrac{1}{2}dR^2)\right)^{1/2} \tag{17}$$

where E_0 (total energy) and P_0 (angular momentum) are arbitrary constants. Differentiating action S with respect to the arbitrary constants E_0 and P_0 gives, respectively,

$$\begin{aligned} z - z_0 &= (\tfrac{1}{2}a)^{1/2}\int \mathrm{d}R[E' - P_0^2/2aR^2 + \tfrac{1}{2}R^2(c' - dR/2)]^{-1/2} \\ \theta - \theta_0 &= \frac{1}{\sqrt{2a}}\int \mathrm{d}R(P_0^2/R^2 + \tfrac{1}{2}b)[E' - P_0^2/2aR^2 + \tfrac{1}{2}R^2(c' - dR^2/2)]^{-1/2} \end{aligned} \tag{18}$$

where $E' \equiv E_0 - bP_0/2a$ and z_0 and θ_0 are initial values. Equation (18) is the general solution of the problem.

The amplitude R oscillates between two limits $R_{\min}$ and $R_{\max}$. For $P_0 = 0$, the solution of equation (18) is expressed in the form:

$$\begin{aligned} R &= A\mathrm{dn}(u, k^2) \\ \frac{\mathrm{d}}{\mathrm{d}z}\theta &= \frac{b}{a} \end{aligned} \tag{19}$$

where dn is a Jacobi elliptic function, $c' \geqslant 0$, $d \leqslant 0$, $0 \geqslant E \geqslant \geqslant 4dc'^2$,

$$A^2 = \frac{-d}{c' + (cc'^2 - 4dE)^{1/2}} \qquad u = B(z - z_0)$$
$$k^2 = \frac{2(c'^2 - 4dE)^{1/2}}{c' + (c'^2 - 4dE)^{1/2}} \qquad B^2 = \frac{c'}{a} + \left[\left(\frac{c'}{a}\right)^2 + 4\left(\frac{E}{a}\right)\right]^{1/2}. \tag{20}$$

In the limit $k \to 1$, i.e. $E \to 0$, we get a localised solution

$$R = A \operatorname{sech} u. \tag{21}$$

For a, c, d, Im $\phi \ll 1$, equation (1) reduces to

$$\frac{\partial}{\partial t} W = \frac{\partial^2}{\partial x^2} W + W - W^3 \tag{22}$$

where $W \equiv \operatorname{Re} \phi$. We seek a solution which is a function of the scaling variable $s = \exp[(1 + \mu^2)t + \mu x]$ (μ being a constant).

Assuming a series solution for a small scaling variable, we have

$$W = \frac{As}{1 + |A|s} \tag{23}$$

where A is a constant and $s = \exp(3/2t + x/\sqrt{2})$. Note that the solution (23) is aperiodic and has a unique propagation velocity.

References

[1] Haken H 1983 *Synergetics* (Berlin: Springer)
[2] Graham R 1975 *Fluctuations, Instabilities and Phase Transitions* ed T Riste (Berlin: Springer)
[3] Katou K and ter Haar D to be published
[4] Kuramoto Y 1983 *Chemical Oscillations, Waves and Turbulence* (Berlin: Springer)
[5] Grimmett G R and Strizaken D R 1982 *Probability and Random Process* (Oxford: Clarendon)
[6] Aronson D G and Weinberger H F 1978 *Adv. Math.* **30** 33
[7] Chen F F 1974 *Introduction to Plasma Physics* (New York: Plenum)
[8] Suzuki M 1981 *Adv. Chem. Phys.* **48** 195